Problem 4-1

We are told that, for every number on first list (call it \(x\)), there is one number on the other list (call it \(y\)) so that the product of the two numbers stays the same. Therefore, the largest number on one list must be paired with the smallest number from the other list, so \(xy = 2 \times 180 = 360\). The number that gets paired with 30 is the value of \(x\) for which \((x)(30) = 360\). That value is \(12\).

[NOTE: This is called an inverse variation.]

Problem 4-2

We are told that at least one root is an integer. The sum of the roots of the first equation is \(a\), and the sum of the roots of the second equation is \(b\). Since both \(a\) and \(b\) are integers, the other root of each equation is also an integer. The product of the roots of the first equation is 2014. The product of the roots of the second equation is 2015. The largest common integral factor of any two consecutive integers is \(1\).

Problem 4-3

When First had gone 10 km, Second had gone 8 km and Third had gone 6 km. Since the ratio of the distances traveled by Second and Third is 8:6 = 4:3, Third traveled \(\frac{3}{4}\) of the distance traveled by Second. When Second finished the 10 km race, Third had gone 7.5 km; so Second won by 2.5 km.

Problem 4-4

The circle’s area is \(\pi\), so its radius is 1. There is a theorem in mathematics that says that of all triangles that can be inscribed in a given circle, the equilateral triangle has the greatest area. To make the longest side of an inscribed triangle as small as possible, all sides must have the same length. Here’s why: any triangle of area \(A\) whose base is shorter than the base of an equilateral triangle of area \(A\) must have legs that are longer than those of the equilateral triangle. The length of one side of the equilateral triangle that is inscribed in a circle of radius 1 is \(\sqrt{3}\).

Problem 4-5

\[ n + \cdots + (n+98) = \frac{99}{2}(n+(n+98)) = \frac{99}{2}(2n+98) = 99(n+49) = (3^2\times11)(n+49) = P. \] The least perfect positive cube value of \(P\) is \(3^3\times11^3\). To find the least such \(n\), we set \((n+49)\) and \(3\times11^2 = 363\) equal to each other. Solving, \(n = 363 - 49 = 314\).

Problem 4-6

The shaded region below the line through \((0,0)\) and \((4,1)\) consists of a square labeled 1 in the diagram plus the lower half of the \(1 \times 4\) rectangle with \((0,0)\) and \((4,1)\) as opposite vertices. The total area below that line is 3, so the line we want must cross the line \(x = 4\) above \((4,1)\). We need a total shaded area of \(9/2 = 4.5\). If the line through the origin crosses \(x = 4\) at \((4,y)\), then \(1 + \frac{bh}{2} = 1 + 4y/2 = 4.5\), so \(y = 7/4\). The slope of the line that connects \((0,0)\) to \((4,7/4)\) is \(\frac{7}{16}\).