Key Fluencies

Grade	Required Fluency
K	Add/subtract within 5
1	Add/subtract within 10
2	Add/subtract within 20 Add/subtract within 100 (pencil and paper)
3	Multiply/divide within 100 Add/subtract within 1000
4	Add/subtract within 1,000,000
5	Multi-digit multiplication
6	Multi-digit division Multi-digit decimal operations
7	Solve $px + q = r$, $p(x + q) = r$
8	Solve simple 2×2 systems by inspection

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Table 1. Common addition and subtraction situations.[3]

	Result Unknown	Change Unknown	Start Unknown
Add to	Two bunnies sat on the grass. Three more bunnies hopped there. How many bunnies are on the grass now? 2 + 3 = ?	Two bunnies were sitting on the grass. Some more bunnies hopped there. Then there were five bunnies. How many bunnies hopped over to the first two? 2 + ? = 5	Some bunnies were sitting on the grass. Three more bunnies hopped there. Then there were five bunnies. How many bunnies were on the grass before? ? + 3 = 5
Take from	Five apples were on the table. I ate two apples. How many apples are on the table now? 5 - 2 = ?	Five apples were on the table. I ate some apples. Then there were three apples. How many apples did I eat? 5 - ? = 3	Some apples were on the table. I ate two apples. Then there were three apples. How many apples were on the table before? ? - 2 = 3

	Total Unknown	Addend Unknown	Both Addends Unknown[4]
Put Together/ Take Apart[5]	Three red apples and two green apples are on the table. How many apples are on the table? 3 + 2 = ?	Five apples are on the table. Three are red and the rest are green. How many apples are green? 3 + ? = 5, 5 - 3 = ?	Grandma has five flowers. How many can she put in her red vase and how many in her blue vase? 5 = 0 + 5, 5 = 5 + 0 5 = 1 + 4, 5 = 4 + 1 5 = 2 + 3, 5 = 3 + 2

	Difference Unknown	Bigger Unknown	Smaller Unknown
Compare[6]	("How many more?" version): Lucy has two apples. Julie has five apples. How many more apples does Julie have than Lucy? ("How many fewer?" version): Lucy has two apples. Julie has five apples. How many fewer apples does Lucy have than Julie? 2 + ? = 5, 5 - 2 = ?	(Version with "more"): Julie has three more apples than Lucy. Lucy has two apples. How many apples does Julie have? (Version with "fewer"): Lucy has 3 fewer apples than Julie. Lucy has two apples. How many apples does Julie have? 2 + 3 = ?, 3 + 2 = ?	(Version with "more"): Julie has three more apples than Lucy. Julie has five apples. How many apples does Lucy have? (Version with "fewer"): Lucy has 3 fewer apples than Julie. Julie has five apples. How many apples does Lucy have? 5 - 3 = ?, ? + 3 = 5

3. Adapted from Boxes 2-4 of *Mathematics Learning in Early Childhood,* National Research Council (2009, pp. 32-33).

4. These take apart situations can be used to show all the decompositions of a given number. The associated equations, which have the total on the left of the equal sign, help children understand that the = sign does not always mean *makes or results in* but always does mean *is the same number as.*

5. Either addend can be unknown, so there are three variations of these problem situations. Both Addends Unknown is a productive extension of this basic situation, especially for small numbers less than or equal to 10.

6. For the Bigger Unknown or Smaller Unknown situations, one version directs the correct operation (the version using *more* for the bigger unknown and using *less* for the smaller unknown). The other versions are more difficult.

Table 2. Common multiplication and division situations.[7]

	Unknown Product $3 \times 6 = ?$	Group Size Unknown ("How many in each group?" Division) $3 \times ? = 18$ and $18 \div 3 = ?$	Number of Groups Unknown ("How many groups?" Division) $? \times 6 = 18$ and $18 \div 6 = ?$
Equal Groups	There are 3 bags with 6 plums in each bag. How many plums are there in all? *Measurement example.* You need 3 lengths of string, each 6 inches long. How much string will you need altogether?	If 18 plums are shared equally into 3 bags, then how many plums will be in each bag? *Measurement example.* You have 18 inches of string, which you will cut into 3 equal pieces. How long will each piece of string be?	If 18 plums are to be packed 6 to a bag, then how many bags are needed? *Measurement example.* You have 18 inches of string, which you will cut into pieces that are 6 inches long. How many pieces of string will you have?
Arrays,[8] Area[9]	There are 3 rows of apples with 6 apples in each row. How many apples are there? *Area example.* What is the area of a 3 cm by 6 cm rectangle?	If 18 apples are arranged into 3 equal rows, how many apples will be in each row? *Area example.* A rectangle has area 18 square centimeters. If one side is 3 cm long, how long is a side next to it?	If 18 apples are arranged into equal rows of 6 apples, how many rows will there be? *Area example.* A rectangle has area 18 square centimeters. If one side is 6 cm long, how long is a side next to it?
Compare	A blue hat costs \$6. A red hat costs 3 times as much as the blue hat. How much does the red hat cost? *Measurement example.* A rubber band is 6 cm long. How long will the rubber band be when it is stretched to be 3 times as long?	A red hat costs \$18 and that is 3 times as much as a blue hat costs. How much does a blue hat cost? *Measurement example.* A rubber band is stretched to be 18 cm long and that is 3 times as long as it was at first. How long was the rubber band at first?	A red hat costs \$18 and a blue hat costs \$6. How many times as much does the red hat cost as the blue hat? *Measurement example.* A rubber band was 6 cm long at first. Now it is stretched to be 18 cm long. How many times as long is the rubber band now as it was at first?
General	$a \times b = ?$	$a \times ? = p$ and $p \div a = ?$	$? \times b = p$ and $p \div b = ?$

7. The first examples in each cell are examples of discrete things. These are easier for students and should be given before the measurement examples.

8. The language in the array examples shows the easiest form of array problems. A harder form is to use the terms rows and columns: The apples in the grocery window are in 3 rows and 6 columns. How many apples are in there? Both forms are valuable.

9. Area involves arrays of squares that have been pushed together so that there are no gaps or overlaps, so array problems include these especially important measurement situations.

Table 5. The properties of inequality.

Here a, b, and c stand for arbitrary numbers in the rational or real number systems.

Exactly one of the following is true: $a < b$, $a = b$, $a > b$.
If $a > b$ and $b > c$ then $a > c$.
If $a > b$, then $b < a$.
If $a > b$, then $-a < -b$.
If $a > b$, then $a \pm c > b \pm c$.
If $a > b$ and $c > 0$, then $a \times c > b \times c$.
If $a > b$ and $c < 0$, then $a \times c < b \times c$.
If $a > b$ and $c > 0$, then $a \div c > b \div c$.
If $a > b$ and $c < 0$, then $a \div c < b \div c$.

The more sophisticated mental operations in mathematics of analysis, synthesis, and evaluation are impossible without rapid and accurate recall of specific knowledge.

Learning the arithmetic facts in the first four grades can be pretty hard work for some students. Learning these facts through only rote memorization will make it more difficult than it needs to be for students. Teaching students strategies to help them learn and memorize the arithmetic facts will make life easier on the kids, their parents, and especially on you.

Before any memorization takes place, the concept for each operation should be explained fully so the students are comfortable in their understanding of the operation.

Addition

Thinking Strategies for Learning the Addition Facts

There are 100 basic arithmetic facts, zero through nine. That's a bunch. But if we use more effective strategies to help students learn, then memorizing these facts will become easier for the students. These strategies have been listed in the order we suggest teachers use to teach their own students.

1. **Adding zero:** Students can quickly grasp the rule for adding zero; the sum is always the other number. $8 + 0 = 8$, $0 + 4 = 4$

2. **Counting on by 1 or 2:** Students can find sums like $5 + 1$ or $6 + 2$ by simply counting on. This thinking strategy allows students to check off 18 of the addition facts. That leaves 63 facts to be learned.

3. **Sums to 5**: Students can learn combinations to 5, such as $3 + 2$ or $4 + 1$. Sums to 5 is redundant.

4. **Sums to 10:** Students can learn combinations to 10, such as $6 + 4$ or $8 + 2$. More facts can be crossed off the list of 100 and we are down to 56 facts to learn.

5. **Doubles:** For whatever reason, students seem to be able to remember the sums of doubles. That might be a consequence of skip counting in earlier grades. The consequence of knowing doubles is another 6 facts can be checked off our list. That leaves 50 facts to learn, we're halfway home.

6. **Adding 10's:** Students can quickly see the pattern develop when adding tens, the units digit stays the same.

7. **Doubles plus one:** This strategy overlaps the other strategies of doubles and counting on by one. While more sophisticated, students should be taught to use this strategy when the addends are consecutive numbers. For instance, 7 + 8 becomes 7 + 7 + 1. Another example, 8 + 9 becomes 8 + 8 + 1.That's seven more off the list.

8. **Doubles plus two:** This method works when the addends differ by two. When this occurs it is possible to subtract 1 from one addend and add one to the other addend. This results in a doubles fact that has already been memorized, 7 + 5 becomes 6 + 6. Another example, 6 + 8 becomes 7 + 7. Some people call the **Doubles Plus 2** strategy **Sharing Doubles** and attack the problem differently. Using the Doubles plus 2 strategy, 6 + 8 becomes 6 + 6 + 2. 7 + 5 becomes 5 + 5 + 2. We now have 31 facts left to learn.

9. **Nines:** It should be pointed out to students that when adding nine, the ones digit in the sum is always one less than the number added to 9. For example 7 + 9 = 16, the 6 is one less than 7. Another example, 5 + 9 = 14. 45 facts to go.

10. **Commutativity:** By changing the order, 3 + 4 to 4 + 3, it should be pointed out that's an additional 21 facts the students now know. That leaves 10 facts to learn. But it's really five because the commutative property can be used on those 10.

There are other strategies to learning the addition facts. Some teachers might use "Combinations to 5" as a strategy. Rather than using Sharing doubles, some teachers might use "Doubles plus 2". That doesn't matter. What does matter is you make sure students understand the meaning of addition and you develop strategies to help them memorize the facts.

It is expected that students respond automatically when asked a basic addition fact.

Subtraction

Thinking Strategies for Learning the Subtraction Facts

1. **Fact families:** This strategy is the most commonly used and works when students understand the relationship between addition and subtraction. When students see 6 – 2 and think 2 + ? =6. However, if this strategy is used with the following strategies, students will find greater success in a shorter period of time.

2. **Counting backwards:** This method is similar to Counting on used in addition. It isn't quite as easy. Some might think if you can count forward, then you can automatically count backward. This is not true –try saying the alphabet backwards. Students should only be allowed to count back **at most** three.

3. **Zeros:** The pattern for subtracting zero is readily recognizable. 5 – 0 = 5

4. **Sames:** This method is used when a number is subtracted from itself; this is another generalization that students can quickly identify. 7 – 7 = 0.

5. **Recognizing Doubles:** Recognizing the fact families associated with adding doubles.

6. **Subtracting tens:** This is a pattern that students can pick up on very quickly, seeing that the ones digit remains the same.

7. **Subtracting from ten:** Recognizing the fact families for Sums to 10.

8. **Subtracting nines:** Again, the pattern that develops for subtracting 9 can be easily identified by most students. They can quickly subtract 9 from a minuend by adding 1 to the ones digit in the minuend. 17 – 9 = 8, 16 – 9 = 7.

9. **Subtracting numbers with consecutive ones digits:** This pattern will always result in a difference of 9, 16 – 7 = 9, 13 – 4,= 9 15 – 6 = 9 all have ones digits that are consecutive and the result is always 9.

10. **Subtracting numbers with consecutive even or consecutive odd ones digits:** This pattern will always result in a difference of 8. 14 – 6 = 8, 13 – 5 = 8, 12 – 4 = 8.

These strategies clearly help students to subtract quickly. How you teach these strategies, allowing the students see the patterns develop, will make students more comfortable using these "shortcuts" and get them off their fingers.

Having said that, as with many of the concepts and skills in math, students need to compare and contrast problems to make them more recognizable to them. Without being able to identify the proper strategy by examining the problem, memorizing these strategies may become more burdensome and cause greater confusion than just rote memorization.

So while you might teach one strategy at a time, as you add to the number of strategies students can use for a specific numbers, you will need to review previous strategies and, this is important, combine strategies on the same work sheets asking students to only identify the strategy they would use for each problem and why they are using it. Being able to compare and contrast will lead to increased student understanding, comfort, and achievement using these strategies..

For example,
16 – 9, students are subtracting 9, they add one to the units digit.
15 – 7, students are subtracting numbers with consecutive odd units digits, the difference is 8.
17 – 8, students are subtracting numbers with consecutive units digits, the difference is 9

Multiplication

Thinking Strategies for Learning the Multiplication Facts

1. **Commutativity:** As with learning the addition facts, order can be changed when learning the multiplication facts. Hence, rather than learning 100 facts, we really only have to learn 55 facts.

2. **Multiplication by zero:** Students can easily grasp that 0 times any number is zero.

3. **Multiplication by one:** Again, the generalization is easy for students to see that 1 times any number is the number.

4. **Multiplication by two:** Students should be taught that multiplying by two is the Doubling strategy used in addition. Using the first four strategies, we have learned 27 more facts; only 28 remain to be learned.

5. **Multiplication by five:** Students can often be taught the fives by referring to the minute hand on a clock.

6. **Squaring:** As with the addition facts, students seem to learn square numbers faster than other facts.

7. **Multiplication by ten:** This pattern is very easy for students to see.

8. **Multiplication by nine:** Patterns emerge when multiplying by 9. One pattern is the sum of the digits in the product is always equal to 9. The other pattern is the ten's digit is always one less than the factor multiplied by 9. $9 \times 6 = 54$. Notice $5 + 4 = 9$ and the 5 in the product is one less 6, the number being multiplied by 9. Another example, 8 x 9 = 72, the sum is 9 and the tens digit is one less than the tens digit. Only 10 facts to go.

9. **Distributive property:** Students should feel comfortable breaking numbers apart and using previously learned information. For instance, 7×6 might be rewritten as $7(5 + 1)$. This would allow a student to use the 7×5 fact that he knows and add that to the 7×1 fact to get 42.

10. **Finger math:** Facing your palms of your hands to your face, let the baby finger become the 6 finger, the ring finger the 7, the middle finger the 8, the index finger be the nine finger. To multiply the 7×7, place the two ring fingers together. The number of fingers touching and below represent the tens digit. There are four fingers, multiply the fingers above those on each hand to determine the ones digit. That would be 3 x 3. The product is 49. Try 7×9 by touching the index finger on one hand with the index finger on the other hand. There are 6 fingers touching or below so the

answer is sixty something and you now multiply the fingers on top to get the ones digit. That's 1 x 3, the final answer is 63.

There are other strategies for learning the multiplication facts. There are two things I want to point out to you before we go any further. First, you don't have students memorize the multiplication facts until you have taught the concept. Make sure the kids feel comfortable in what they are learning. Second, notice we have not taught the facts sequentially. We taught them in an order to help students learn so students experience a sense of accomplishment.

Division

11. **Fact families:** This strategy works when students understand the relationship between multiplication and division. When students see 24 ÷ 6, they have to relate that to 6 × ? = 24.

Using the fact strategies presented, students learn the easier facts first that provides successful experiences that build confidence and motivate students to learn more. It's important that students not only understand each operation, but they get off their fingers as quickly as possible. Memorizing the basic arithmetic facts simplifies the process of recalling and allows that information to become automatic. Understanding and critical thought can then be built on this base of knowledge.

Division by zero undefined. The reason that division by zero is not allowed is because of how division is defined: a/b = c if and only if a = bc. Let's look at an example of that definition, 8/2 = 4 if and only if 8 = 2 x 4. That's true. Now let's try dividing by zero. 8/0 = #, where # represents any number. 8/0 = # if and only if 8 = 0 x #. Well zero times any number will never result in 8, therefore since this is not true, it does not fit the definition of division. Therefore, we are not allowed to divide by zero.

Standards for Mathematical Practice for Parents

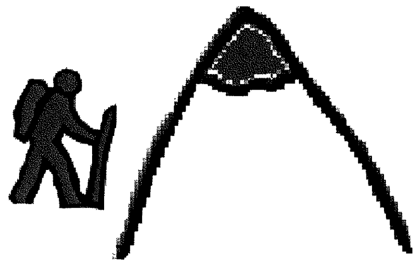
Practice Standard	How a child can use the practice standards	Questions to ask
1. Make sense of problems and persevere in solving them.	• I can make my own plan for solving the problem and stick with it even if it is difficult. • I can check the reasonableness of my answer. • I can solve it a second way to make sure I am right!	• What plan can you make to solve this problem? • Can you draw a picture or act out the problem? • What information is in the problem and what are you trying to figure out?
2. Reason abstractly and quantitatively.	• I can use numbers and words to help make sense of problems. • I can think about what each number represents. • I can think about the relationships between the numbers in the problem. • I can think about what property might be used to solve the problem. • I can think about whether other operations might be used.	• Can you explain what the numbers in the problem mean? • How did you decide to use this operation?
3. Construct viable arguments and critique the reasoning of others.	• I can explain my thinking using objects, drawings or actions • I can consider the thinking of other students • I can ask questions to clarify my understanding • I can make connections to other strategies	• How could you prove that.....? • How can we be sure? • Is this like another problem you have solved before?
4. Model with mathematics.	• I can recognize math in everyday life and use it to solve problems. • I can use pictures, words, objects or symbols to solve. • I can use number lines, arrays or other models to help myself as I solve the problem or to represent my solution.	• What model could you construct that might help you solve this problem? • Can you visualize the action in this problem?

5. Use appropriate tools strategically.	• I can use math tools such as number lines, calculators, objects, tables, etc. to solve a problem. • I can use estimates when problem solving.	• What tools could we use to solve this problem? • What information do you have that might help?
6. Attend to precision.	• I can be careful when I use math and clear when I share my ideas. • I always think about whether my answer is reasonable! • I try to be efficient and concise when I solve a problem. (this looks different at various grade levels) • I can test my solution by solving a different way or by modeling the solution and checking for reasonableness.	• How do you know your solution is reasonable? • How could you test your solution to see if it accurately answers the problem?
7. Look for and make use of structure.	• I can see and understand how numbers and shapes are put together as parts and wholes. • I look for patterns that can help me solve a problem. • I think about other problems I have solved before and whether they can help me with this problem. • I try to connect mathematical ideas.	• What do you notice when…? • What patterns do you find in…? • What are some other problems that are similar to this one?
8. Look for and express regularity in repeated reasoning.	• I can notice when calculations are repeated and use these ideas to create a strategy. • I think about whether patterns are always true in all situations. • I can create rules for patterns.	• Is this always true? • What do you notice about…? • What is happening in this situation?

Mathematical Practices

I can ...

1. Make sense of problems and persevere in solving them

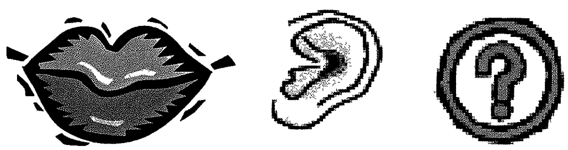


2. Reason abstractly and quantitatively

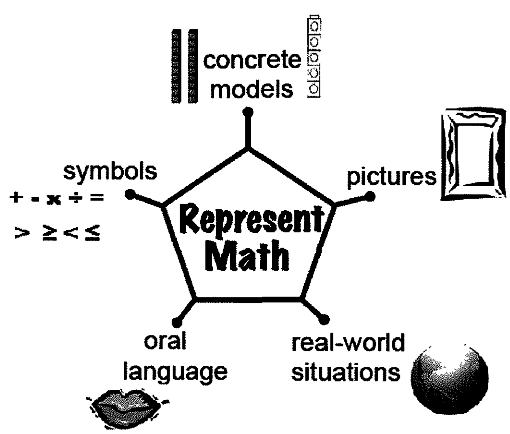
a b c . . .

1 2 3 . . .

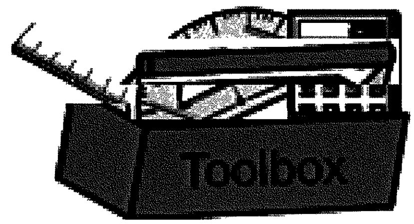
3. Construct viable arguments & critique the reasoning of others



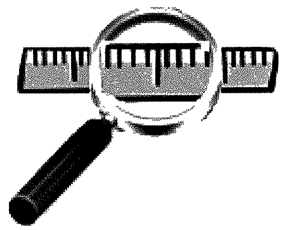
4. Model with mathematics



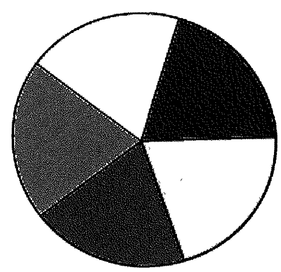
5. Use appropriate tools Strategically



6. Attend to precision



7. Look for and make sense of Structure



8. Look for & express regularity in repeated reasoning