Contest # 3

Answers & Solutions 12/8/15

Problem 3-1

To maximize the largest angle, minimize the two smallest angles. If the two smallest angles have measures of 1 and 1, the measure of the largest angle of the triangle will be 178° or 178̊.

Problem 3-2

The first output is \(\frac{1492}{2015}\). This output is fed back in and the second output is 1492 divided by \(\frac{1492}{2015}\). That output is 2015.

Problem 3-3

Method I: By inspection, the sequence is \(a, a, 2a, 4a, 8a, \ldots\), so for \(n > 2\), each term is twice the preceding term. Therefore, is the 100th term is 2015, the 101st term is 4030.

Method II: We'll let \(S_n\) represent the sum of the first \(n\) terms of the sequence, and \(a_n\) represent its \(n\)th term. We're told that \(a_n = S_{n-1}\), so 2015 = \(a_{100} = S_{99}\), so
\[
a_{101} = a_1 + \ldots + a_{99} + a_{100} = S_{99} + a_{100} = 2015 + 2015 = 4030.
\]

Problem 3-4

In each isosceles triangle, let each vertex angle be \(x\) and let each base angle be \(y\). In each triangle, \(x^\circ + 2y^\circ = 180^\circ\). In the diagram, the 12 triangles surround the center, so 10\(x\) + 2\(y\) = 360°. Subtracting the first equation from the second, 9\(x\) = 180°, so \(x = 20\°\).

Problem 3-5

Method I: From the binomial theorem, \((\sqrt{2} - 1)^5 = (\sqrt{2})^5 - 5(\sqrt{2})^4 + 10(\sqrt{2})^3 - 10(\sqrt{2})^2 + 5\sqrt{2} - 1 = 4\sqrt{2} - 5(4) + 10(2\sqrt{2}) - 10(2) + 5\sqrt{2} - 1 = 29\sqrt{2} - 41 = \sqrt{1682} - \sqrt{1681}\), so \(k = 1681\).

Method II: Use the table feature on your calculator, using \(y = \sqrt{x+1} - \sqrt{x}\). Use “ask” for the “independent variable” and “auto” for the “dependent variable.” By underestimating and overestimating, you should be able to zero in on \(x = 1681\) in not too many tries.

Problem 3-6

Each pair \((m,n)\) has the form \((t^2, t^3)\), where \(t\) is a positive integer. Since \(m + n = t^2 + t^3 = t^2(t + 1)\) is a square, \(t + 1\) must also be a square. Let \(t + 1 = k^2\). [Conversely, if \(m = t^2\) and \(n = t^3\), where \(t = k^2 - 1\), then \(m + n = t^2 + t^3 = (1 + t)(t^2) = k^2(k^2 - 1)^2 = (k^3 - k)^2\) is a square.] Since \(m = t^2, t = k^2 - 1,\) and \(m < 1000,\) it follows that \((k^2 - 1)^2 < 1000.\) Taking square roots, \(k^2 - 1 < 31.6 \ldots,\) so \(k^2 < 32.6 \ldots.\) The largest such \(k^2\) is 25, so \(t + 1 = k^2 = 25\) and \(t = 24.\)
Finally, \(m = t^2 = 24^2 = 576.\)
[Note: \(m + n = 24^2 + 24^3 = 576 + 13824 = 14400 = 120^2.\)]