Investigation: Exploring Properties of Plane Shapes

Coordinate methods can be used to reason about plane shapes. Given two points (x_1, y_1) and (x_2, y_2), the distance between them can be calculated and the midpoint can be located. These methods are useful to explore plane shapes and their properties. As you work on the following problems, look for an answer to this question:

How can coordinates be used to reason about plane shapes?

1. Triangle ABC is an acute triangle with coordinates A(2,4), B(8,8) and C(17,1).
   a. Graph ∆ABC and calculate the perimeter of the triangle.
   b. In order to calculate the area of the triangle, the height is needed. Graph point D(7,3) on side AC and draw the line segment BD. The segment BD represents the height from vertex B to the base side AC. Calculate the area of ΔABC and compare your result with others. Resolve any differences.

2. In activity 1, the coordinates for point D were given so that BD represented the height of the triangle. BD is also called the **altitude** of the triangle. Consider how the altitude might be determined if the point were not given.
   a. What is the relationship between the altitude and the base side?
   b. Which of the following figures illustrates an altitude of the triangle?

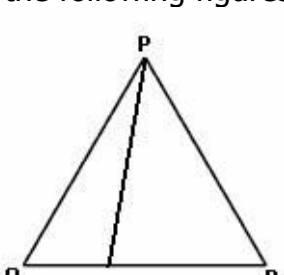


Figure 1

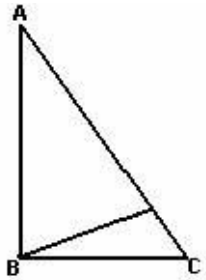


Figure 2

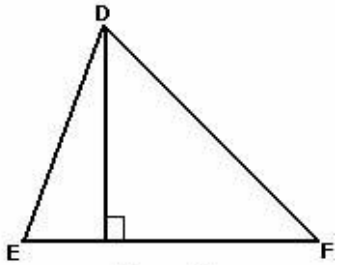
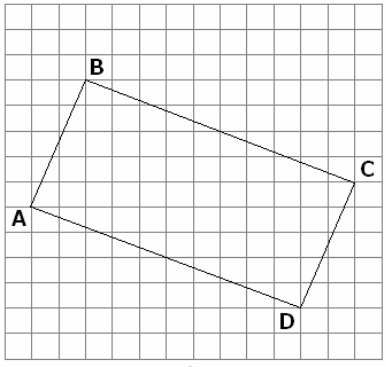


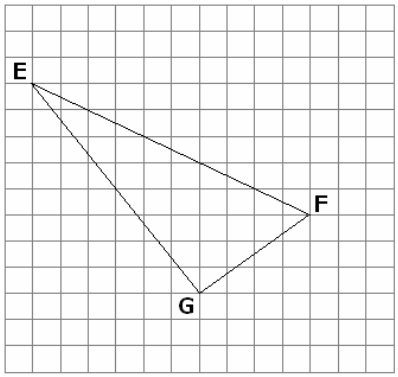
Figure 3

   c. Refer back to the triangle in activity 1. Calculate the slope of side AC and the slope of altitude BD. Record the slopes in reduced fraction form. Describe any relationship between the two slopes.

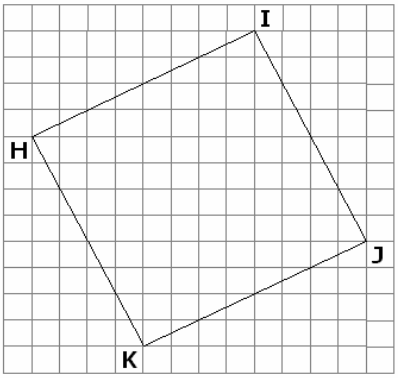
3. Right angles (angles measuring 90°) can be found in plane shapes such as right triangles, rectangles, and squares. Two lines are considered perpendicular when they intersect at a right angle. For each plane shape, identify a pair of perpendicular sides and compare their slopes. Describe the relationship between the slopes of perpendicular lines.



rectangle ABCD

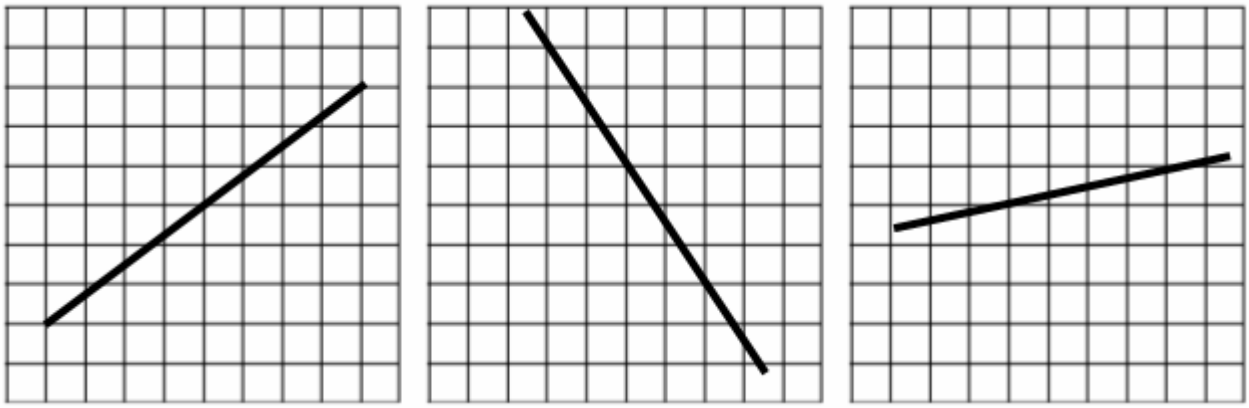


right triangle EFG



square HIJK

4. In activities 2 & 3, you discovered that two intersecting lines are perpendicular if and only if their slopes are opposite reciprocals. Below are three line segments. Use the fact that the slopes of perpendicular lines are opposite reciprocals to draw a line segment perpendicular to the given line segment. In each case, justify that the lines are perpendicular.



5. Quadrilateral KLMN is a rectangle with coordinates K(-3,-3), L(0,9), M(4,8) and N(?,?).
   a. Find the coordinates for point N. Explain how you determined the coordinates.
   b. Verify that quadrilateral KLMN is a rectangle by giving evidence related to its sides and angles.
   c. What do you notice about the sides that are opposite of one another? What can be said about the sides if the slopes are the same?

6. Lengths and slopes of the sides of plane shapes can assist in identifying the specific type of shape. Coordinates of different triangles and quadrilaterals are given below. In each case, carefully draw the figure on a coordinate grid and determine as precisely as possible the type of shape. You may want to refer to *Shapes and Their Properties* handout for specific properties of different shapes.
   a. A(1,2) B(2,5) C(5,7) D(4,4)
   b. E(4, -1) F(5,6) G(1,3)
   c. H(-3,2) I(-2, 6) J(2,7) K(1,3)
   d. L(2,2) M(5,-2) N(9,1) P(6,5)

7. Coordinate points are often used to define the endpoints for a line segment. Linear equations can be used to define lines that extend in both directions without end. Examine the following linear equations.

   A. $y = 8 + 3x$ B. $y = 2x + 5$ C. $y = \frac{-1x}{3} + 7$ D. $y = \frac{6 - 4x}{-2}$ E. $y = -3 + 5x$

   a. Identify a pair of parallel lines and explain how you know they are parallel by examining the equations.
   b. Identify a pair of perpendicular lines and explain how you know they are perpendicular by examining the equations.
   c. Write the equation of the line parallel to $y = 3x + 12$ and passing through (6,13).
   d. Write the equation of the line perpendicular to $y = \frac{2}{3}x - 5$ and passing through (4,9).

Summarize the Mathematics

Coordinate methods can be used to determine characteristics of plane figures.

a) Discuss what calculations are necessary to determine if
   i. two line segments are the same length
   ii. two lines are parallel
   iii. two lines are perpendicular

b) Given a line $y = \frac{g}{h}x + b$ and a point (x_1, y_1) discuss how to write an equation of a line that is
   i. parallel to the given line
   ii. perpendicular to the given line

Check Your Understanding

Consider quadrilateral QRST with coordinates Q(1,0) R(2,3) S(5,2) and T(5,-3).

a) Graph quadrilateral QRST.
b) Determine as precisely as possible the name of quadrilateral QRST. Justify your reasoning using appropriate calculations.
c) What appears to be true about the diagonals of quadrilateral QRST? Use the coordinates and appropriate calculations to explore your conjecture.

Investigation: Exploring Properties of Plane Shapes (Teacher Notes)

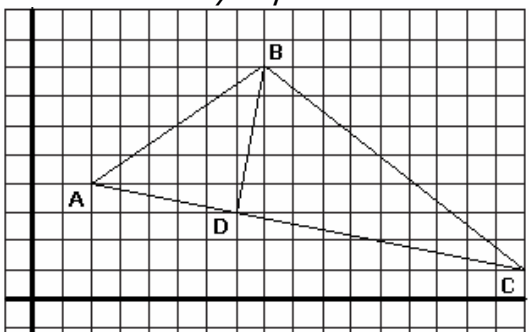
NC CCSS Math I: Expressing Geometric Properties with Equations

G-GPE.4 Use coordinates to prove simple geometric theorems algebraically. For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; ~~prove or disprove that the point (1, √3) lies on the circle centered at the origin and containing the point (0, 2).~~

G-GPE.5 Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).

G-GPE.7 Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula.

1. a. *Students may express answers in radical or decimal approximate.*



length AB = $\sqrt{(8-2)^2+(8-4)^2} = \sqrt{52} = 2\sqrt{13} \approx 7.2$ units

length BC = $\sqrt{(8-17)^2+(8-1)^2} = \sqrt{130} \approx 11.4$ units

length CA = $\sqrt{(17-2)^2+(1-4)^2} = \sqrt{234} = 3\sqrt{26} \approx 15.3$ units

perimeter of ΔABC is $2\sqrt{13}+\sqrt{130}+3\sqrt{36} \approx 33.9$ units

b. length BD = $\sqrt{(8-7)^2+(8-3)^2} = \sqrt{26} \approx 5.1$ units

area of ΔABC = $\frac{1}{2}(3\sqrt{26})(\sqrt{26}) = 39$ square units

2.

a. The altitude and the base side are perpendicular, meaning that they meet at a 90° angle.

b. Figure 3

c. slope AC = $\frac{-3}{15} = \frac{-1}{5}$ and slope BD = 5. The slopes are opposite reciprocals. *Students may or may not use the vocabulary 'opposite' and 'reciprocal'.*

3. *Students are asked to identify one pair of perpendicular sides so they are not expected to list all pairs. Also, the images are provided without the coordinate axes. Students may determine the slope by comparing the vertical change to the horizontal change from one point to another point* $\left(\frac{rise}{run}\right)$*. In activity 6, students can explore using the slope formula to calculate slope.*

rectangle ABCD: Pairs of sides which are perpendicular are AB & BC, BC & CD, CD & DA, and DA & BA. The slopes of AB and DC are $\frac{5}{2}$ and slopes of BC and AD are $\frac{-2}{5}$.

right triangle EFG: Pair of sides which are perpendicular is EG & GF. The slope of EG is $\frac{-4}{3}$ and the slope of GF is $\frac{3}{4}$.

square HIJK: Pairs of sides which are perpendicular are HI & IJ, IJ & JK, JK & KH, and KH & HI.

The slope of HI and KJ is $\frac{1}{2}$ and the slope of IJ and KH is -2.

The slopes of perpendicular lines are opposite reciprocals.

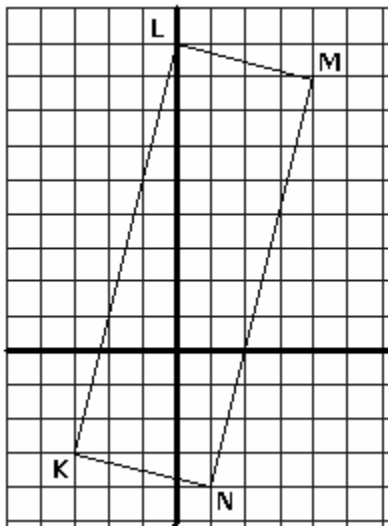
4.

The slope of the first line is $\frac{3}{4}$ so any line with a slope of $\frac{-4}{3}$ will be perpendicular.

The slope of the second line is $\frac{-3}{2}$ so any line with a slope of $\frac{2}{3}$ will be perpendicular.

The slope of the third line is $\frac{1}{4}$ so any line with a slope of -4 will be perpendicular.

5.



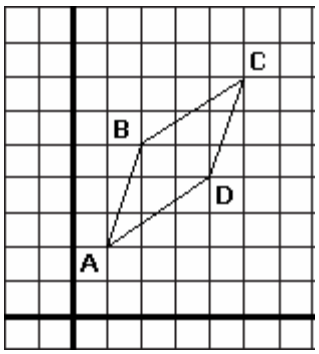
a. The coordinates for N are (1, -4). To find the coordinates for N, make the line KN perpendicular to LK. Since the slope of LK is 4 then the slope of KN is $\frac{-1}{4}$. Also, MN must be perpendicular to LM. Since the slope of LM is $\frac{-1}{4}$ then the slope of MN is 4. Where MN and KN intersect is the coordinate for N. *Some students may use slopes are the same for parallel lines and give a similar argument.*

b. The slopes of the sides that intersect at a common vertex are opposite reciprocals (4 and $\frac{-1}{4}$) so the lines are perpendicular meaning the figure has all right angles. The sides opposite have the same length (KN and LM have a length of $\sqrt{17} \approx 4.1$ and LK and MN have a length of $\sqrt{153} \approx 12.4$. Thus, the figure is a rectangle.

c. The sides opposite of one another have the same length and are parallel. If the slopes are the same for two distinct lines then the lines are parallel.

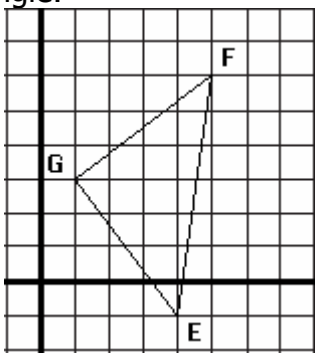
6.

1. Quadrilateral ABCD is a parallelogram. The opposite sides have the same length. Also, since the opposite sides have the same slope they are parallel.



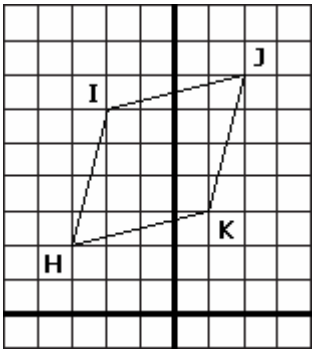
	length	slope
AB	$\sqrt{10}$	3
BC	$\sqrt{13}$	$\frac{2}{3}$
CD	$\sqrt{10}$	3
DA	$\sqrt{13}$	$\frac{2}{3}$

2. Triangle EFG is an isosceles right triangle.
Sides FG and GE are both 5 units long. Also, their slopes are opposite reciprocals which means the lines are perpendicular forming a 90° angle.



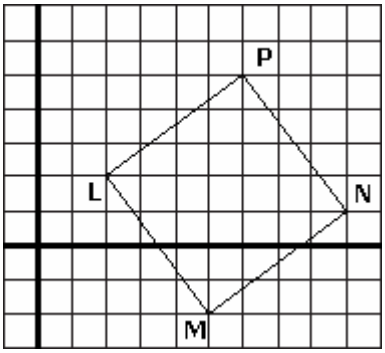
	length	slope
EF	$\sqrt{50}$	7
FG	5	$3/4$
GE	5	$-4/3$

3. Quadrilateral HIJK is rhombus.
All sides have the same length. Also, opposite sides have the same slope which means that opposite sides are parallel.



	length	slope
HI	$\sqrt{17}$	4
IJ	$\sqrt{17}$	$1/4$
JK	$\sqrt{17}$	4
KH	$\sqrt{17}$	$1/4$

4. Quadrilateral LMNP is a square.
All sides have the same length. Also, the slopes of the sides adjacent to one another are opposite reciprocals. This means that they are perpendicular forming a 90° angle.



	length	slope
LM	5	$3/4$
MN	5	$-4/3$
NP	5	$3/4$
PL	5	$-4/3$

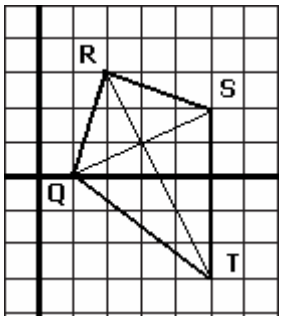
7.
a. B and D both have a slope of 2 so the lines are parallel.
b. A and C have slopes that are opposite reciprocals so the lines are perpendicular.
c. y=3x – 5
d. $y = \frac{-3}{2}x + 15$

Summarize the Mathematics
a) To determine if two lines
i. have the same length you should use the distance formula.
ii. are parallel the slopes should be the same.
iii. are perpendicular the slopes should be opposite reciprocals.

b) The line that is parallel should have the same slope. To find the y intercept, start with the y_1 value and subtract the slope $\frac{g}{h}$ x_1 number of times. For the line that is perpendicular the slope should be the opposite reciprocal. To find the y intercept, start with the y_1 value and subtract the slope $\frac{-h}{g}$ x_1 number of times.

Check Your Understanding

a)



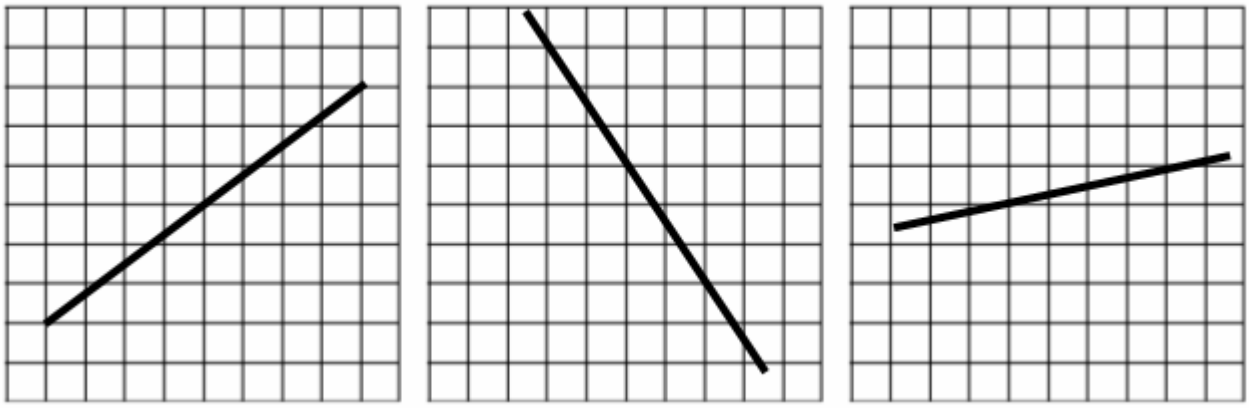
	length	slope
QR	$\sqrt{10}$	$1/3$
RS	$\sqrt{10}$	$-1/3$
ST	5	undefined
TQ	5	$-3/4$

b) Quadrilateral QRST is a kite. The adjacent sides QR and RS have a length of $\sqrt{10}$ units. The other pair of adjacent sides ST and TQ have a length of 5 units.

c) The slope of RT is -2 and the slope of QS is ½ so the diagonals are perpendicular since the slopes are opposite reciprocals. The equation of the line that includes segment RT is y=-2x+7. The midpoint of QS is (3,1) and is on the line y=-2x+7. This makes RT the perpendicular bisector of QS. *Students may recognize that RT goes through the midpoint and justify it by referring to the graph.*

Investigation: Exploring Properties of Plane Shapes

Problem #4



Shapes and Their Properties

Triangle	Must prove
Isosceles	Two sides are congruent
Right	Two sides are perpendicular

Quadrilateral	Must Prove
Trapezoid	Two sides are parallel
Kite	Two sets of adjacent sides are congruent
Parallelogram	Opposite sides are congruent Opposite sides are parallel
Rectangle	Opposite sides are congruent Opposite sides are parallel Adjacent sides are perpendicular
Rhombus	All sides congruent Opposite sides parallel
Square	All sides congruent Opposite sides parallel Adjacent sides are perpendicular