Contest Number 6

March 11, 2014

Name __________________ Teachers ___________________ Grade Level ___ Score ___

Time Limit: 30 minutes

FINAL CONTEST OF THE YEAR

6-1. What are both values of \( x \) that satisfy \( x^2 - x = 2^2 - 2 \)?

6-2. Circles of radii 1, 2, and 3 are externally tangent to each other and internally tangent to a circle of radius 6, as shown. The quadrilateral connecting the centers of these four circles is a rectangle. What is the perimeter of this rectangle?

6-3. In how many different ways can 3 men and 3 women line up in a row so that no two people of the same gender stand adjacent to each other?

6-4. What are both values of \( x \) for which \( \sqrt{\log x} \), \( \sqrt{\log 3} \), and \( \sqrt{\log 4} \) could be the lengths of the sides of a right triangle?

6-5. Line \( \ell \) is perpendicular to lines \( m \) and \( n \). If the product of the slopes of all 3 lines is \( -8 \), what is the slope of line \( \ell \)?

6-6. In this problem, brackets will denote the greatest integer function, so that \( [n] \) is the greatest integer \( \leq n \). For example, \( [\pi] = 3 \). What value of \( x \) satisfies \( x^{[x]} = \sqrt[4]{2014} \)? [NOTE: For this problem, your answer must be exact. A decimal approximation is NOT acceptable.]

Eighteen books of past contests, Grades 4, 5, & 6 (Vols. 1, 2, 3, 4, 5, 6), Grades 7 & 8 (Vols. 1, 2, 3, 4, 5, 6), and HS (Vols. 1, 2, 3, 4, 5, 6), are available, for $12.95 each volume ($15.95 Canadian), from Math League Press, P.O. Box 17, Tenafly, NJ 07670-0017.

Problem 6-1

Since $x^2 - x = 2^2 - 2 = 2$, $x^2 - x - 2 = (x-2)(x+1) = 0$; so $x = \frac{2-\sqrt{4}}{2} = 1$.

Problem 6-2

In the diagram at the right, the circle with radius 6 has been removed, and the 3 externally tangent circles are shown. As you can see in the diagram, the length of each side of the rectangle is the sum of the lengths of the radii of two of the circles. The perimeter of the rectangle is $2(3+4) = 14$.

NOTE: Proving that the center of the circle to which the other three are internally tangent has its center on the circle with radius 3 is very interesting. Go to http://www.ams.org/samplings/feature-column/fcarc-kissing for an excellent American Mathematical Society discussion of “kissing circles.”

Problem 6-3

The first position can be occupied by any of 6 people. The second position must be filled by one of the 3 people of a different gender than the first person, and so on. The total number of different ways to line up these six people is $6 \times 3 \times 2 \times 2 \times 1 \times 1 = 2(3!)^2 = 2(6^2) = 72$.

Problem 6-4

The longest of the three given lengths must be the hypotenuse, so the side with length $\sqrt{\log 3}$ cannot be the hypotenuse. Applying the Pythagorean Theorem, either $\log x + \log 3 = \log 4$ (so $\log x = 1$, or $x = e^1$), or $\log 3 + \log 4 = \log x$ (so $12 = x$). The two values of $x$ are $\frac{4}{3}, 12$.

Problem 6-5

Since $\ell$ and $m$ are perpendicular, $(\text{slope of } \ell)(\text{slope of } m) = -1$. Since the product of all three slopes is $-8$, the slope of $n$ must be 8. Since $\ell$ and $n$ are perpendicular, the product of their slopes is $-1$, from which we can conclude that the slope of $\ell = \frac{-1}{8}$.

Problem 6-6

$2^2 < \sqrt{2014} = x^2 < 3^2$, so $2 < x < 3$. Thus, $[x] = 2$, so the equation we’re solving is $x^2 = \sqrt{2014}$. Solving, $x = \left(\frac{2014}{1/2}\right)^{1/2} = \sqrt{2014}$. 

Contests written and compiled by Steven R. Conrad & Daniel Flegler ©2014 by Mathematics Leagues Inc.