11.5 The Vector Equation of a Line in Cartesian 2-Space

Consider the line \( \ell \) which passes through the point A with position vector \( a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \) and which is parallel to the vector \( v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \). Let \( r = \begin{pmatrix} x \\ y \end{pmatrix} \) be the position vector of a general point P on \( \ell \).

Since AP is parallel to v, \( \overrightarrow{AP} = tv \) where \( t \) is a scalar.

Thus \( \overrightarrow{OP} - \overrightarrow{OA} = tv \) or \( r = a + tv \).

The equation \( r = a + tv \) or \( \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + t \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \) is the vector equation of the line \( \ell \).

The conversion of the vector equation to the normal Cartesian equation is illustrated in the following example.

**Example** Write the vector equation \( \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + t \begin{pmatrix} 3 \\ -2 \end{pmatrix} \) in the form \( ax + by = c \).

\[
\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + t \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 2+3t \\ -1-2t \end{pmatrix}
\]

which gives \( x = 2 + 3t \) and \( y = -1 - 2t \).

Making \( t \) the subject of each equation gives \( t = \frac{x-2}{3} \) and \( t = \frac{y+1}{-2} \). Hence

\[
\frac{x-2}{3} = \frac{y+1}{-2}
\]

and so \(-2x + 4 = 3y + 3\) or \(2x + 3y = 1\).
Motion of a Body Moving in a Straight Line in Cartesian 2-Space

In the following work, the vector components each represent a displacement of 1 unit of distance either in the direction of the $x$-axis or in the direction of the $y$-axis.

A body moves in a straight line in the direction of the vector $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. If the body starts from point A with position vector $\mathbf{a}$ at time $t = 0$, the position vector of the body at any subsequent time $t$ is given by $\mathbf{r} = \mathbf{a} + t\mathbf{v}$.

The vector $\mathbf{v}$ is called the velocity vector of the body and the length of this vector, $|\mathbf{v}|$, denotes the (constant) speed of the body.

**Example**

A body initially at the point $(2, 1)$ has a velocity vector $3\mathbf{i} - 4\mathbf{j}$. If the distance unit is a metre and the time unit is a second, find:

(a) the position vector of the body after 8 seconds;
(b) the speed of the body.

The position vector of the body after $t$ seconds is given by $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + t(3\mathbf{i} - 4\mathbf{j})$.

(a) When $t = 8$, $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + 8(3\mathbf{i} - 4\mathbf{j}) = 26\mathbf{i} - 31\mathbf{j}$.
(b) The speed of the body is $|3\mathbf{i} - 4\mathbf{j}| = 5 \text{ m s}^{-1}$.

**Example**

If a body moving in the direction of the vector $7\mathbf{i} + 24\mathbf{j}$ where the component unit is a kilometre and the speed of the body is $200 \text{ m s}^{-1}$, find the body's velocity vector.

Let $\mathbf{v} = k(7\mathbf{i} + 24\mathbf{j})$. Then $|\mathbf{v}| = k\sqrt{7^2 + 24^2} = 25k$. But $|\mathbf{v}| = 200$, so $k = 8$.

Therefore the velocity vector is $\mathbf{v} = 8(7\mathbf{i} + 24\mathbf{j}) = 56\mathbf{i} + 192\mathbf{j}$. 
1. Find a vector equation for the straight line passing through the point \( A \) and parallel to the vector \( v \) in the following: \( A = (2, -5) \) and \( v = -7i + 3j \)

2. Find the vector equation for the line joining the following pairs of points: \((4, -2)\) and \((9, 8)\)

3. Find a Cartesian equation in the form \( Ax + By = C \) for question #1

4. Find a vector which is parallel to the following line: \( 3x - 4y = 5 \)

5. Find a Cartesian equation for a line which passes through the point \( A \) and which is perpendicular to the vector \( v \), in the following: \( A = (2, 3) \) and \( v = 3i + j \)

6. Find a Cartesian equation for a line which passes through the point \( A \) and which is parallel to the vector \( v \), in the following: \( A = (-1, 2) \) and \( v = 2i + 3j \)

7. At time \( t \) seconds, the position vector \( r \) of a moving particle is given by:

\[
\begin{pmatrix}
3 \\
-1
\end{pmatrix} + t \begin{pmatrix}
-5 \\
12
\end{pmatrix}
\]

(Displacement unit = 1 metre).

A. What is the initial position vector of the particle?

B. Find the (constant) speed of the particle.

C. Show that the particle passes through the point with position vector \(-49.5i + 125j\). At what time is this?
ANSWERS:

<table>
<thead>
<tr>
<th></th>
<th>1. ( \vec{r} = (2\vec{i} - 5\vec{j}) + t(-7\vec{i} + 3\vec{j}) ) or ( \vec{r} = (2, -5) + t(-7, 3) )</th>
<th>2. ( \vec{r} = (4\vec{i} - 2\vec{j}) + t(5\vec{i} + 10\vec{j}) ) or ( \vec{r} = (4, -2) + t(5, 10) )</th>
<th>3. ( 3x + 7y = -29 )</th>
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<tbody>
<tr>
<td></td>
<td>4. ( \vec{v} = \begin{pmatrix} 4 \ 3 \end{pmatrix} ) or ( \vec{v} = 4\vec{i} + 3\vec{j} )</td>
<td>5. ( 3x + y = 9 )</td>
<td>6. ( 3x - 2y = -7 )</td>
</tr>
<tr>
<td>7A.</td>
<td>( \vec{v} = 3\vec{i} - \vec{j} )</td>
<td>7B. ( 13 \frac{\text{meters}}{\text{second}} )</td>
<td>7C. ( t = 10.5 \text{ seconds} )</td>
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