Contest # 4 Answers & Solutions 1/8/13

Problem 4-1

When a line crosses the y-axis, its x-coordinate is 0, so \( a + 2013 = 0 \) and \( a = -2013 \).

Problem 4-2

Method I: Since the length of a diagonal of the \( 12 \times 16 \) rectangle is 20, the length of a diagonal of the smaller rectangle is \( 20 - (5 + 5) = 10 \). By similar triangles, the smaller rectangle's sides are half as long as the larger rectangle's sides. Therefore, the area of the smaller rectangle is \( \frac{12}{2} \times \frac{16}{2} = 6 \times 8 = 48 \).

Method II: Since the smaller rectangle's diagonals are half as long as those in the larger rectangle, and since these two rectangles are also similar, the area \( A \) of the smaller rectangle is \( \left( \frac{1}{2} \right)^2 = \frac{1}{4} \) times the area of the larger rectangle, so \( A = \frac{1}{4} \times 16 \times 12 = 48 \).

Problem 4-3

Two days ago, \( x \) dogs skated. The number of dogs skating yesterday was 20% more than that, so it was \( \frac{6}{5} \). Today, we had 40% more dogs skating than yesterday. The number of dogs skating today (which must be an integer) is \( \frac{7}{5} \) of \( \frac{6}{5} \), or \( \frac{42x}{25} \). This will be an integer whenever \( x \) is a multiple of 25. The least positive integral value of \( x \) is 25.

Problem 4-4

This question really asks how many integers < 100 are rational powers of 8. Recall that \( 2 = 8^{1/3} \), and no other prime is a rational power of \( 8^a \). If we raise 8 to any non-negative multiple of 1/3, the result will be an integer. Since \( x = 8^{k/3} \) is a positive integer < 100 if and only if \( k = 0, 1, 2, 3, 4, 5, \) and 6, there is a total of 7 values.

[NOTE: Here's a proof: Since \( x \) is a positive integer, it follows that \( \log_{8} x > 0 \). If \( \log_{8} x = \frac{u}{v} \) where \( u \) and \( v \) are integers with \( u \geq 0 \) and \( v > 0 \), then \( 8^{u/v} = x \), or equivalently \( 2^{3u} = x^v \). By prime factorization, \( x = 2^k \), where \( k \) is an integer. If \( x = 2^k \), then \( \log_{8} x = \frac{k}{3} \), where \( k/3 \) is a rational number. Since \( 0 < x < 100 \), the only solutions are \( k = 0, 1, 2, 3, 4, 5, \) and 6.]

Problem 4-5

Draw the isosceles trapezoid and its circumcircle, as shown to the right. Draw a perpendicular from the center of the circle to both bases of the trapezoid. This bisects both of those bases, since a line through the center of a circle and perpendicular to a chord bisects the chord. Draw the radii shown in the diagram. Using the Pythagorean Theorem in both right triangles that have the radii as hypotenuses, we get \( x^2 + 9^2 = r^2 = (x+4)^2 + 5^2 \). Solving, \( x = 5 \). The area of the circle is \( \pi r^2 = \frac{106}{\pi} \).

Problem 4-6

Since \( P(1) = P(2) = P(3) = P(4) = P(5) = 0 \), we have \( P(x) = k(x-1)(x-2)(x-3)(x-4)(x-5) \), where \( k \neq 0 \) is a real number. If we expand \( P(x) \), the lead term will be \( kx^5 \). Since \( kx^5 = x^5 \), we know that \( k = 1 \). Now evaluate \( P(-1) \): \( P(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + e \), so \( P(-1) = (-1)^5 + a(-1)^4 + b(-1)^3 + c(-1)^2 + d(-1) + e = -1(-a+b+c+d+e) \). From line 2 above, \( P(-1) = (-2)(-3)(-4)(-5)(-6) = -720 \), so \( -a+b+c+d+e = 720 \).

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