Contest # 1

Answers & Solutions

Problem 1-1
The ratio of the circumference of the circle to the perimeter of the square is \(\frac{2\pi r}{4r} = \frac{\pi}{2}\).

Problem 1-2
The equation \(|a-b| = 1005\) means that, on a number line, the distance between points \(a\) and \(b\) is 1005. Similarly, since \(|b-c| = 1007\), the distance between points \(c\) and \(b\) is 1007. If \(a\) and \(c\) are on the same side of \(b\), then \(a\) and \(c\) are 2 units apart. If \(a\) and \(c\) are on opposite sides of \(b\), then the distance between \(a\) and \(c\) would be 1005 + 1007 = 2012. Therefore, the two possible values of \(|a-c|\) are 2, 2012.

Problem 1-3
The length of any side of a polygon must be less than half the perimeter. For example, in a triangle with perimeter 100, the sides can have lengths 25, 26, and 49. In a quadrilateral with perimeter 100, the sides can have lengths 17, 17, 17, and 49. No matter how many sides a polygon has, the longest side's length cannot equal or exceed the sum of the lengths of all the remaining sides. The maximum length is 49.

Problem 1-4
Every positive prime except 2 is odd. The difference between any two odd numbers is even, so two primes can differ by an odd number only if one of the primes is 2. Begin by looking for differences between 2 and odd primes: \(3-2 = 1\); \(5-2 = 3\); \(7-2 = 5\); \(11-2 = 9\). So, 7 is the smallest unattainable odd number. Let's see which even numbers are expressible as a difference between odd primes: \(5-3 = 2\); \(7-3 = 4\); \(11-5 = 6\). This proves that the least positive integer we cannot write as the difference between positive primes is 7.

Problem 1-5
Each non-empty subset of \(\{1,2,3,4,5,6,7,8,9\}\) that contains 2 or more digits has one such ordering. The total number of non-empty subsets of a 9-element set is \(2^9-1\). Since this total includes 9 one-digit subsets (which represent 9 one-digit numbers), we must subtract 9, making the answer \(2^9-1-9 = 502\).

Problem 1-6
Since \(x\) is the number of coins given to each boy, and \(y\) is the number of coins given to each girl, if \(T\) is the total number of coins, then \(T = 3x + 4y\). We want the largest \(T\) for which there is only one solution in positive integers \((x,y)\). First, let's look at an example of how we can get a second solution when a first solution is known: In this case, by either 1) adding 4 to the value of \(x\) while subtracting 3 from the value of \(y\), or 2) subtracting 4 from the value of \(x\) while adding 3 to the value of \(y\), we're merely adding and subtracting 12 coins, so the value of \(T\) does not change. Thus, since \((1,5)\) is a solution when \(T = 23\), the pair \((5,2)\) is a second solution. If \(x_0 > 4\) and \((x_0,y_0)\) satisfies \(T = 3x + 4y\), then \((x_0-4,y_0+3)\) also satisfies \(T = 3x + 4y\), with \(x_0-4 > 0\). Similarly, if \(y_0 > 3\) satisfies \(T = 3x + 4y\), then \((x_0+4,y_0-3)\) also satisfies \(T = 3x + 4y\), with \(y_0-3 > 0\). Therefore, \((4,3)\) is the solution with the largest possible values of \(x\) and \(y\) for which no second solution pair exists. In this case, \(T = 24\).